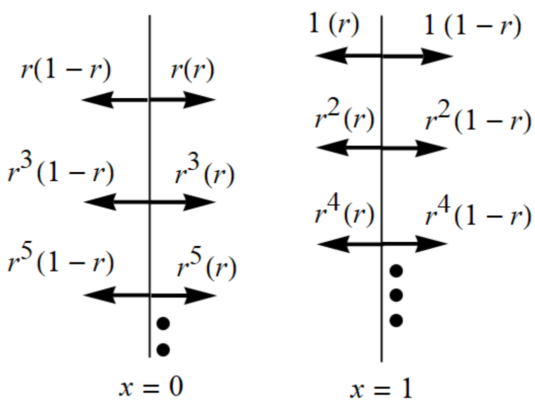
Problem 16

Suppose a large number of particles are bouncing back and forth between $x = 0$ and $x = 1$, except that at each endpoint some escape. Let r be the fraction reflected each time; then $(1 - r)$ is the fraction escaping. Suppose the particles start at $x = 0$ heading toward $x = 1$; eventually all particles will escape. Write an infinite series for the fraction which escape at $x = 1$ and similarly for the fraction which escape at $x = 0$. Sum both the series. What is the largest fraction of the particles which can escape at $x = 0$? (Remember that r must be between 0 and 1.)

[TYPOS: Change "are" to "is," change "escape" to "escapes," and change "escape" to "escapes."]

Solution

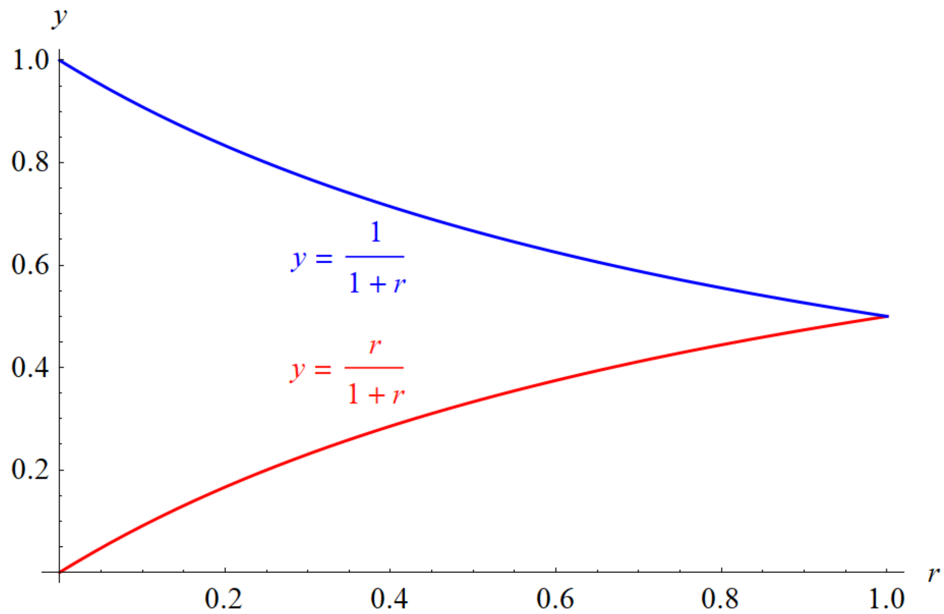
The schematic below illustrates the fraction of particles that reflects and passes through each time a wall is hit.



At the start the fraction of particles present is 1. $1(r)$ reflects back, and $1(1 - r)$ passes through. Of the r of particles, $r(r)$ reflects back, and $r(1 - r)$ passes through. This continues indefinitely.

Fraction of Particles Passing Through $x = 0$	Fraction of Particles Passing Through $x = 1$
$\sum_{n=0}^{\infty} r^{2n+1}(1-r)$	$\sum_{n=0}^{\infty} r^{2n}(1-r)$
$\sum_{n=0}^{\infty} (r^2)^n r(1-r)$	$\sum_{n=0}^{\infty} (r^2)^n (1-r)$
$r(1-r)\sum_{n=0}^{\infty} (r^2)^n$	$(1-r)\sum_{n=0}^{\infty} (r^2)^n$
$r(1-r)\left[\frac{1}{1-(r^2)}\right]$	$(1-r)\left[\frac{1}{1-(r^2)}\right]$
$r(1-r)\left[\frac{1}{(1+r)(1-r)}\right]$	$(1-r)\left[\frac{1}{(1+r)(1-r)}\right]$
$\frac{r}{1+r}$	$\frac{1}{1+r}$

Below is a plot of these two functions versus r. Note that $r \neq 1$.



The largest fraction of particles that can leave through $x = 0$ is $1/2$ as $r \to 1$.